A nonlinear Cauchy-Poisson problem initiated by an impulsive surface pressure

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The Cauchy-Poisson problem is a classical initial value problem of water waves. There are two types of Cauchy-Poisson problems: (i) An initial surface deflection is released from rest. (ii) A flat initial surface is forced into motion by an pressure impulse that delivers momentum to the fluid.

We will here consider the second type of Cauchy-Poisson problem. This fully nonlinear free-surface problem is investigated analytically by a small-time expansion

$$(p, \varphi, \eta) = (p_{-1}, 0, 0)\,\delta(t) + H(t)\,((p_0, \varphi_0, 0) + t\,(p_1, \varphi_1, \eta_1) + t^2\,(p_2, \varphi_2, \eta_2) + \ldots). \qquad (1)$$

Here p is the pressure, φ the velocity potential and η is the surface elevation.The incompressible semi-infinite fluid has an initially horizontal free surface ($\eta_0 = 0$) and is forced into motion by a given surface pressure impulse $P(x) = p_{-1}(x,0)$, which is continous in x. We consider here the function

$$P(x) = P_0\,(1 + (x/L)^2)^{-2}, \qquad (2)$$

where P_0 is the amplitude of the surface pressure impulse. Figure 1 shows the chosen function $P(x)/P_0$ as a function of x/L.

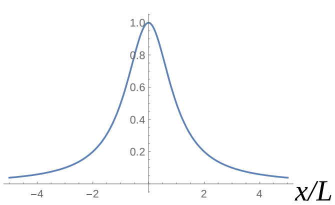

Figure 1

Multipole-type flows are generated by this type of surface pressure impulse distribution. The solution is expressed in dimensionless variables, where the unit of length L is defined by the imposed pressure impulse $P(x)$ given above. The unit of dimensionless time is $\rho L^2/P_0$, where ρ is the density of the liquid. There is one dimensionless parameter, which is the dimensionless gravity G defined as

$$G = \rho\, Lg/P_0\,, \qquad (3)$$

where g is the gravitational acceleration. Gravity enters as a linearized term in the third-order solution for the surface elevation. The present nonlinear solution is of interest only for a dimensionless gravity G of order 1 and smaller. If gravity is stronger than that, this Cauchy-Poisson problem will follow linear theory fairly well throughout the process. We are interested in flows where nonlinearity has time to develop before dispersion reduces the importance of nonlinearity as time goes. The nonlinearities that develop for dimensionless time smaller than one, will leave their marks on the later flow, even after the flow has become linearized for greater times.

The full nonlinear problem will be solved to third order in the surface elevation. In Figure 2 we show the total second-order solution, $\eta(x,t) = t\,\eta_1(x) + t^2\,\eta_2(x)$, given by a brown graph, and compare it with the first-order solution $t\,\eta_1(x)$, given by a blue graph. The chosen dimensionless time is $t=0.25$. Here the asymptotic series is still a good approximation to the exact nonlinear solution, but this asymptotic series will break down already at $t=0.4$.

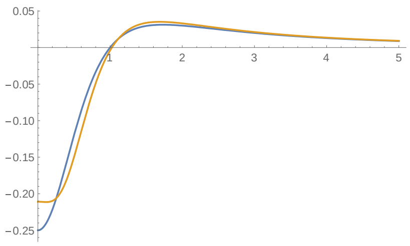

Figure 2

The dimensionless form of these solutions are given by

$$\eta_1 = (x^2 - 1)(1 + x^2)^{-2}, \quad 8\,\eta_2(x) = (5 - 80\,x^2 + 50\,x^4 + 8\,x^6 + x^8)(1 + x^2)^{-5}, \qquad (4)$$

Work is progressing where also an asymmetric surface pressure distribution is considered, in combination with the one shown here. The full nonlinear interactions to third order in η will be calculated.